can be improved. First, the code is written in such a way that eight N-vectors must be stored in addition to the two N-vectors which define the table being interpolated. By a simple rewriting of the code, the eight N-vectors can be reduced to one. Second, the algorithm of Ref. 4 sets up the linear system in terms of the second derivatives at the nodal points. Then, both the second and third derivatives are used in the interpolation computations. If the first derivatives at the nodal points are needed, additional computations would have to be made. An algorithm can be formulated such that the solution of the linear system yields the first derivatives at the nodal points and that this is the only N-vector which needs to be stored along with the two defining the table. The remainder of this Note presents the equatons for such an algorithm.

Consider the cubic polynomial for the kth interval where the k subscripted quantities are evaluated at the beginning of the interval and the $k+1$ subscripted quantities are evaluated at the end of the interval. Let t denote the value of the independent variable within the kth interval for which the interpolated value of the dependent variable y is desired. Let A, B, C, and D represent the constant coefficients of the cubic polynomial in this interval, and write the following expressions:

$$\begin{aligned} y &= A+B(t-t_k)+(C/2)(t-t_k)^2+(D/6)(t-t_k)^3 \\ y' &= B+C(t-t_k)+(D/2)(t-t_k)^2 \\ y'' &= C+D(t-t_k) \end{aligned} \tag{1}$$

In terms of the notation $h = t-t_k$, $H_k = t_{k+1}-t_k$, and $R = h/H_k$, the expressions which relate the constant coefficients of the cubic to the values of the dependent variable and its first derivative at the ends of the interval are given by

$$\begin{aligned} A &= y_k \\ B &= y'_k \\ C &= (6/H_k^2)(y_{k+1}-y_k)-(2/H_k)(y'_{k+1}+2y'_k) \\ D &= (-12/H_k^3)(y_{k+1}-y_k)+(6/H_k^2)(y'_{k+1}+y'_k) \end{aligned} \tag{2}$$

These values may now be substituted into the expressions for the dependent variable and its derivatives to produce the following relations:

$$\begin{aligned} y &= y_k+(3R^2-2R^3)(y_{k+1}-y_k)+ \\ &\qquad H_k(R-2R^2+R^3)y'_k+H_k(-R^2+R^3)y'_{k+1} \\ y' &= (6/H_k)(R-R^2)(y_{k+1}-y_k)+ \\ &\qquad (1-4R+3R^2)y'_k+(-2R+3R^2)y'_{k+1} \\ y'' &= (6/H_k^2)(1-2R)(y_{k+1}-y_k)+ \\ &\qquad (1/H_k)(-4+6R)y'_k+(1/H_k)(-2+6R)y'_{k+1} \end{aligned} \tag{3}$$

The equations written in this form assure continuity of the dependent variable and its first derivative at the ends of the interval as may be demonstrated by using $R = 0$ and $R = 1$ in the above equations to obtain values at the beginning of the kth interval and at the end of kth interval. In order to assure continuity of the second derivative between intervals, the value of y'' at the end of the kth interval is equated to the value of y'' at the beginning of the $k+1$st interval. This yields the following recursive relationship for y'_k:

$$\begin{aligned} H_k y'_{k-1}+2(H_{k-1}+H_k)y'_k+H_{k-1}y'_{k+1} &= \\ 3(H_{k-1}/H_k)(y_{k+1}-y_k)&+3(H_k/H_{k-1})(y_k-y_{k-1}) \end{aligned} \tag{4}$$

The output of the Runge-Kutta-Fehlberg numerical integration process is the table of values of y_k and t_k. For N stepping points (including the first) used by the numerical integrator, N values of y'_k must be determined to construct the cubic spline. The recursive relations of Eq. (4) provide N-2 equations in y'_k. The input of y'_1 and y'_N, the values at the beginning and end of the integration interval, provides a solvable set of equations in y'_k. If conditions are imposed on y''_1 and y''_N, the corresponding conditions on the first derivatives can be obtained from the third of Eqs. (3).

The successive over-relaxation iterative method for solving the linear system is defined by the relation

$$y'_k = y'_k + 1.0717968\Delta_k \tag{5}$$

where

$$\Delta_k = \frac{1}{2(H_{k-1}+H_k)}\left[3\frac{H_{k-1}}{H_k}(y_{k+1}-y_k)+3\frac{H_k}{H_{k-1}}(y_k-y_{k-1})- H_k y'_{k-1}-H_{k-1}y'_{k+1}\right]-y'_k \tag{6}$$

To start the iterative process, the values of y'_k corresponding to a quadratic spline are used; that is,

$$y'_k = \frac{H_{k-1}}{H_k+H_{k-1}}\frac{1}{H_k}(y_{k+1}-y_k)+\frac{H_k}{H_k+H_{k-1}}\frac{1}{H_{k-1}}(y_k-y_{k-1}) \tag{7}$$

The iterative process is continued until the change in the values of y'_k at each nodal point between iterations satisfies a prescribed tolerance which should be correlated with the tolerance prescribed for the integrator. Convergence of the iterative method is assured since the coefficient matrix of the linear system is tridiagonal and irreducibly diagonally dominant. Finally, once the values of y'_k are known, interpolated values of y can be computed from the first of Eqs. (3).

From the previous relations, it is apparent that the only quantities which need to be stored during the computation process are t_k, y_k, and y'_k. The values of H_k, H_{k-1}, and Δk can be computed at each nodal point and do not need to be stored as vectors.

The use of variable-step integration with numerical optimization methods which iterate on variable histories requires interpolation. Furthermore, the interpolation method must be generated accurately and must use minimal storage. The procedure suggested here has a guaranteed accuracy and requires the minimum amount of storage.

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Torsional Vibrations and Stability of Thin-Walled Beams on Continuous Elastic Foundation

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Nomenclature

A = area of cross section of the beam
A_i = constants ($i = 1\ldots4$)
a_i, B_n = constants ($i = 0\ldots4$; $n = 1, 2, \ldots, \infty$)
C_s = torsion constant

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C_w = warping constant
E = modulus of elasticity
G = shear modulus
I_p = polar moment of inertia of beam
K_s = torsional foundation modulus
K = warping parameter
L = length of the beam
N = mode number
P = axial compressive load
σ = P/A, axial compressive stress
p_n = natural frequency of vibration
t = time
T = kinetic energy
z = distance along the length of the beam
Z = z/L, nondimensional distance along the length of the beam
V = potential energy
λ^2 = $\rho I_p L^4 p_n^2/EC_w$, frequency parameter
Δ^2 = $\sigma I_p L^2/EC_w$, load parameter
Δ_{cr}^2 = critical buckling load parameter
γ^2 = $K_s L^4/4EC_w$, foundation parameter
ϕ = angle of twist of the beam
$X(Z)$ = normal function of angle of twist
α, β = positive real quantities
δ = variation operator
ρ = mass density of the material of the beam

Introduction

STATIC and dynamic analysis of beams on elastic foundation occupies a prominent place in contemporary structural mechanics. The vibrations and buckling of continuously supported finite and infinite beams resting on elastic foundation have applications in the design of highway pavements and aircraft runways, and in the use of metal rails for rail road tracks. Very large numbers of studies have been devoted to this subject[1,2] and valuable practical methods for the analysis of such beams have been worked out. Free torsional vibrations and stability of doubly-symmetric thin-walled beams of open section are investigated by Gere,[3] Aggarwal and Cranch,[4] Krishna Murty and Joga Rao,[5] Kameswara Rao, Apparao and Sarma,[6] and Timoshenko and Gere.[7] Recently free torsional vibrations of restrained elastic thin-walled beams is also investigated by Christieno and Salmela.[8] In all the preceding investigations which deal with free vibrations, the effects of elastic foundation and in-plane axial compressive force are not included. The purpose of the present investigation is to include these effects and to study their influence on the natural frequencies and buckling loads of simply supported, fixed and simply supported-fixed thin-walled beams of open section.

Formulation and Analysis

Neglecting the effects of longitudinal inertia and shear deformation, the total potential energy of the beam V, consisting of the strain energy of deformation of the beam, the work done by the external compressive load and the reaction offered by the elastic foundation, is given by[9]

$$V = \frac{1}{2}\int_0^L [EC_w(\phi'')^2 + (GC_s - \sigma I_p)(\phi')^2 + K_s(\phi)^2]\, dz \quad (1)$$

where primes denote differentiation with respect to z. The kinetic energy is

$$T = \frac{1}{2}\int_0^L \rho I_p(\dot{\phi})^2\, dz \quad (2)$$

where dot denotes differentiation with respect to t. Applying Hamilton's principle which states

$$\delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \quad (3)$$

and carrying out necessary variations and integrations we get

$$(GC_s - \sigma I_p)\phi'' - EC_w\phi'''' - K_s\phi = \rho I_p \ddot{\phi} \quad (4)$$

which is the governing differential equation of motion. The natural boundary conditions obtained are

$$\phi''\delta\phi' \Big|_0^L = 0 \quad (5)$$

and

$$[EC_w\phi''' - (GC_s - \sigma I_p)\phi']\,\delta\phi \Big|_0^L = 0 \quad (6)$$

From Eqs. (5) and (6), for the beam simply supported at both ends, the boundary conditions to be satisfied are

$$\phi = \phi'' = 0 \quad \text{at} \quad Z = 0 \quad \text{and} \quad Z = 1 \quad (7)$$

where

$$Z = z/L \quad (8)$$

For the beam fixed at both ends, the boundary conditions are

$$\phi = \phi' = 0 \quad \text{at} \quad Z = 0 \quad \text{and} \quad Z = 1 \quad (9)$$

For the beam simply supported at one end and fixed at the other, the boundary conditions are

$$\phi = \phi'' = 0 \quad \text{at} \quad Z = 0 \quad \text{and} \quad \phi = \phi' = 0 \quad \text{at} \quad Z = 1 \quad (10)$$

For sinusoidal vibration, we take

$$\phi(Z, t) = X(Z)e^{ipt} \quad (11)$$

Using Eqs. (8) and (11), Eq. (4) can be written as

$$X'''' - (K^2 - \Delta^2)X'' - (\lambda^2 - 4\gamma^2)X = 0 \quad (12)$$

where

$$K^2 = L^2 GC_s/EC_w, \quad \Delta^2 = \sigma I_p L^2/EC_w \quad (13)$$

and

$$\gamma^2 = K_s L^4/4EC_w, \quad \lambda^2 = \rho I_p L^4 p_n^2/EC_w \quad (14)$$

The general solution of Eq. (12) is

$$X(Z) = A_1 \cosh \alpha Z + A_2 \sinh \alpha Z + A_3 \cos \beta Z + A_4 \sin \beta Z \quad (15)$$

where

$$\alpha^2 = (1/2)\{(K^2 - \Delta^2) + [(K^2 - \Delta^2)^2 + 4(\lambda^2 - 4\gamma^2)]^{1/2}\} \quad (16)$$

and

$$\beta^2 = (1/2)\{-(K^2 - \Delta^2) + [(K^2 - \Delta^2)^2 + 4(\lambda^2 - 4\gamma^2)]^{1/2}\} \quad (17)$$

From Eqs. (16) and (17), the relation between α and β is

$$\alpha^2 = \beta^2 + (K^2 - \Delta^2) \quad (18)$$

The four arbitrary constants, A_i ($i = 1 \ldots 4$) in Eq. (15) can be determined so as to satisfy the particular boundary conditions of the problem. For any beam there will be two boundary conditions at each end and these four conditions determine the frequency equation. Solving the frequency equation then determines the principal frequencies of vibration and hence the mode shapes.

Applying the boundary conditions given by Eqs. (7, 9, and 10), the following frequency equations are obtained:

$$\sin \beta = 0 \quad (19)$$

$$2\alpha\beta(1 - \cosh \alpha \cos \beta) + (\alpha^2 - \beta^2)\sinh \alpha \sin \beta = 0 \quad (20)$$

and

$$\tanh \alpha = (\alpha/\beta)\tan \beta \quad (21)$$

for the simply supported, fixed, and simply supported-fixed beams, respectively.

From Eq. (19), we have $\beta = N\pi$, and using Eq. (17), the expression for the natural frequency parameter λ, for the simply supported beam is obtained as

$$\lambda = \{N^2\pi^2(N^2\pi^2 + K^2 - \Delta^2) + 4\gamma^2\}^{1/2} \quad (22)$$

By putting $\lambda = 0$, and $N = 1$, in Eq. (22), the expression for the buckling load parameter Δ_{cr}, in this case can be obtained as

$$\Delta_{cr}^2 = [\pi^2 + K^2 + (4/\pi^2)\gamma^2] \quad (23)$$

The frequency equations for the fixed and simply supported-fixed beams given by Eqs. (20) and (21) can be observed to be highly transcendental and can be solved in conjunction with Eqs. (16–18) by lengthy trial-and-error procedure.

In an attempt to derive approximate but satisfactory expressions for the frequency parameter λ and buckling load parameter Δ for the fixed beam, the normal function $X(Z)$ is assumed in the form

$$X(Z) = \sum_{n=1}^{\infty} B_n(1 - \cos 2n\pi Z) \quad (24)$$

which satisfies the boundary conditions given by Eq. (9).

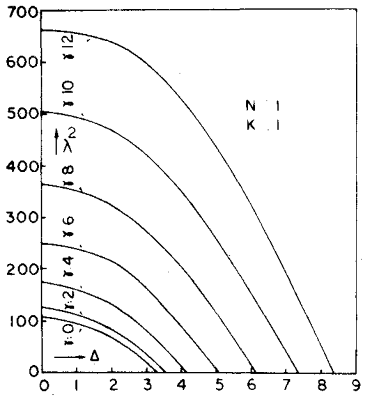

Fig. 1 Values of frequency and buckling parameters for a simply supported beam.

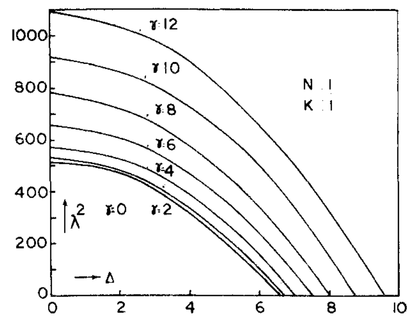

Fig. 2 Values of frequency and buckling parameters for a fixed-fixed beam.

Substituting Eq. (24) in Eq. (12) and performing the Galerkin method of integrating the resulting expression over the whole length of the beam, the expression for the frequency parameter λ is obtained as

$$\lambda = 2\{(N^2\pi^2/3)(4N^2\pi^2 + K^2 - \Delta^2) + \gamma^2\}^{1/2} \qquad (25)$$

In arriving at Eq. (25), only one term of the infinite series of Eq. (24) is utilized. Eq. (25) gives an upper bound for the natural frequency parameter λ for the fixed-end beam as it is obtained by the approximate method due to Galerkin. But it is quite handy and can give values within engineering accuracy.

By putting $\lambda = 0$, and $N = 1$, in Eq. (25), the expression for the buckling load parameter Δ_{cr}, for the clamped beam can be obtained as

$$\Delta_{cr}{}^2 = [4\pi^2 + K^2 + (3/\pi^2)\gamma^2] \qquad (26)$$

For the simply supported-fixed beam, the approximate expressions obtained, for the fundamental frequency parameter λ ($N = 1$), and buckling load parameter Δ_{cr}, by the Galerkin method assuming a power series of the type

$$X(Z) = \sum_{i=0}^{4} a_i Z^i \qquad (27)$$

are

$$\lambda = [238.739 + 11.3686(K^2 - \Delta^2) + 4\gamma^2]^{1/2} \qquad (28)$$

and

$$\Delta_{cr}{}^2 = [21 + K^2 + 0.352\gamma^2] \qquad (29)$$

Out of the five constants a_i ($i = 0, 1, 2, 3, 4$) in Eq. (27), four constants as ratios of the fifth constant can be obtained by utlizing the boundary conditions of Eq. (10) for this case, and the fifth arbitrary constant cancels out in the Galerkin integral.

In the limiting case of the absence of elastic foundation, i.e., $\gamma = 0$, and the compressive load, $\Delta = 0$, all the approximate expressions are observed[9] to be in complete agreement with those derived previously by Gere[3] and Timoshenko and Gere.[7]

Conclusions

Results for the torsional frequency parameter λ, for the first mode ($N = 1$), for the simply supported and fixed beams, obtained from Eqs. (22) and (25) are plotted in Figs. 1 and 2 respectively, for various values of foundation parameter γ and load parameter Δ. The warping parameter is kept at $K = 1$. The values of the critical buckling loads for various values of γ can also be obtained from the graphs for $\lambda = 0$ (i.e., on the axis on which Δ is taken). When the axial load is not present the values of the frequency parameter λ for various values of γ can be obtained for values of $\Delta = 0$ (i.e., on the vertical axis on which λ is plotted). The combined influence of the foundation parameter γ and the load parameter Δ can be observed from the graphs, to be opposing each other. Independently, as the load parameter Δ increases, the frequency parameter λ drops to zero. In the absence of the axial load, the frequency parameter λ increases for increasing values of the foundation parameter γ. Hence the combined influence is the superimposition of the individual effects on the frequency of vibration.

It can be observed from Eqs. (22) and (25) that the influence of foundation parameter γ decreases for increasing values of N (i.e., for higher modes). It is interesting to see from Eq. (22) that for the simply supported beam, for the limiting condition $\gamma = 0.5N\pi\Delta$, the combined influence of elastic foundation and axial compressive load becomes zero. In the case of clamped beam this limiting condition, from Eq. (25), is $\gamma = 0.574N\pi\Delta$.

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Vibration—Stability Relationships for Conservative Elastic Systems

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Introduction

LURIE[1] observed that, for linearly elastic systems, the square of the lateral frequency is "practically" linearly related to the end thrust. According to Southwell's theorem[2] this straight

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